University of North Georgia
Department of Mathematics

Instructor: Berhanu Kidane

Course: Precalculus Math 1113

Text Books: For this course we use free online resources:

See the folder Educational Resources in Shared class files

1) http://www.stitz-zeager.com/szca07042013.pdf (**Book1**)
2) Trigonometry by Michael Corral (**Book 2**)

Other online resources:

E– Book: http://msenux.redwoods.edu/IntAlgText/

Tutorials: http://www.wtamu.edu/academic/anns/mps/math/mathlab/col_algebra/index.htm

- http://archives.math.utk.edu/visual.calculus /
- http://www.ltcconline.net/greenl/java/index.html
- http://en.wikibooks.org/wiki/Trigonometry
- Animation Lessons: http://flashytrig.com/intro/teacherintro.htm
- http://www.sosmath.com/trig/trig.html

For more free supportive educational resources consult the syllabus

Trigonometric Identities

(Book 2 page 65)

Objectives: By the end of this section student should be able to

- Identify Fundamental or Basic Identities
- Find trigonometric values using the trig Identities
- Evaluate trigonometric functions

Identities

Equations: Three types

1) **Conditional equations**: These types of equations have finitely number of solutions.
   Example: a) $2x - 5 = 7x$, b) $3x^2 - 4x - 6 = 0$
2) **Contradictions**: These are equations that do not have solutions
   Examples: $2x - 1 = 2(x - 1) + 6$
3) **Identities**: These types of equations hold true for any value of the variable
   Examples: $(x + 5)(x - 5) = x^2 - 25$

Trigonometric Identities are identities of the Trigonometric equations. We use an identity to give an expression a more convenient form. In calculus and all its applications, the trigonometric identities are of central importance.

Fundamental or Basic Trigonometric Identities

Reciprocal Identities, Quotient Identities and Pythagorean Identities

1) Reciprocal identities (Page: 65)

$$\sin\theta = \frac{1}{\csc\theta}, \quad \text{and} \quad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta}, \quad \text{and} \quad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta}, \quad \text{and} \quad \cot\theta = \frac{1}{\tan\theta}$$

Proof: Follows directly from the definition of trig functions**.**

2) Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \text{and} \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Proof: Follows directly from the definition of trig functions.

3) Pythagorean Identities (page: 66 & 67)

a) $\sin^2\theta + \cos^2\theta = 1$

b) $1 + \tan^2\theta = \sec^2\theta$

c) $1 + \cot^2\theta = \csc^2\theta$

Proof: a) Let $P(x, y)$ be on the terminal side of the angle θ.

Then $r = \sqrt{x^2 + y^2}$ which implies that $r^2 = x^2 + y^2$, $\sin\theta = \frac{y}{r}$, and $\cos\theta = \frac{x}{r}$

And so, $\sin^2\theta + \cos^2\theta = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$

Example: Book 2: Example 3.1, 3.2, 3.3, 3.4, 3.5, 3.6 and 3.7 reading (67 – 69)

Note:

- From **a)** it follows that: $\sin^2\theta = 1 - \cos^2\theta$ and $\cos^2\theta = 1 - \sin^2\theta,$
- **b)** and **c)** are similarly proved.
- $\sin^2\theta$, "**sine squared theta**", means $(\sin\theta)^2$

Example 1:

a) Express $\sin\theta$ in terms of $\cos\theta$

b) Express $\cos\theta$ in terms of $\sin\theta$

c) Express $\tan\theta$ in terms of $\cos\theta$, where θ in Quadrant II

d) If $\tan\theta = \frac{3}{2}$ and θ is in Quadrant III, find $\sin\theta$ and $\cos\theta$

e) If $\cos\theta = \frac{1}{2}$ and θ is in Quadrant IV, find all other trig values of θ

f) Use the basic trigonometric identities to determine the other five values of the trigonometric functions given that $\sin\alpha = 7/8$ and $\cos\alpha > 0$.

g) x is in quadrant **II** and $\sin x = 1/5$. Find $\cos x$ and $\tan x$.

Example 2: Prove the **Pythagorean Identities b)** and **c)**

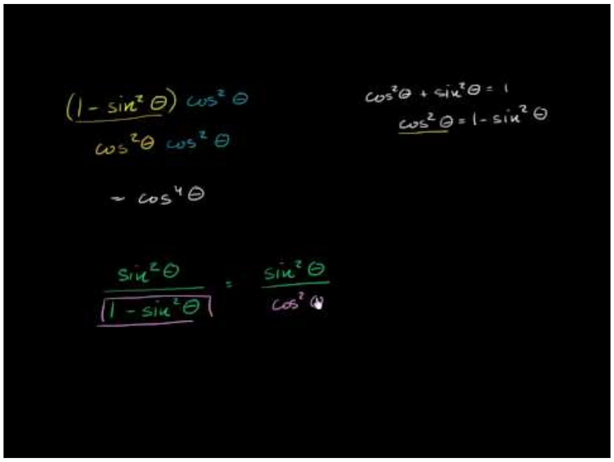
Example 3: Homework Reading page 67 – 69 Examples 3.1, 3.2, **3.3**, 3.4, 3.5, 3.6 and 3.7

(Book 2) Homework Exercises 3.1 page 70: # 1 – 21 odd numbers

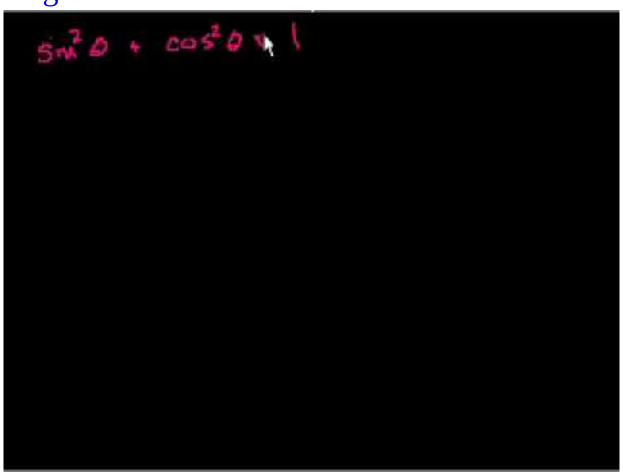
Examples YouTube Videos Trigonometric Identities

1) https://www.youtube.com/watch?v=iKWGv0xcuCA
2) https://www.youtube.com/watch?v=QGk8sYck_ZI
3) https://www.youtube.com/watch?v=raVGSdfBVBg

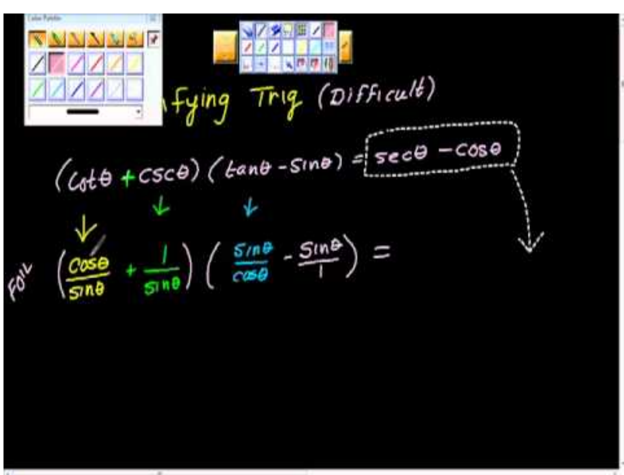
4) Pythagorean Identity



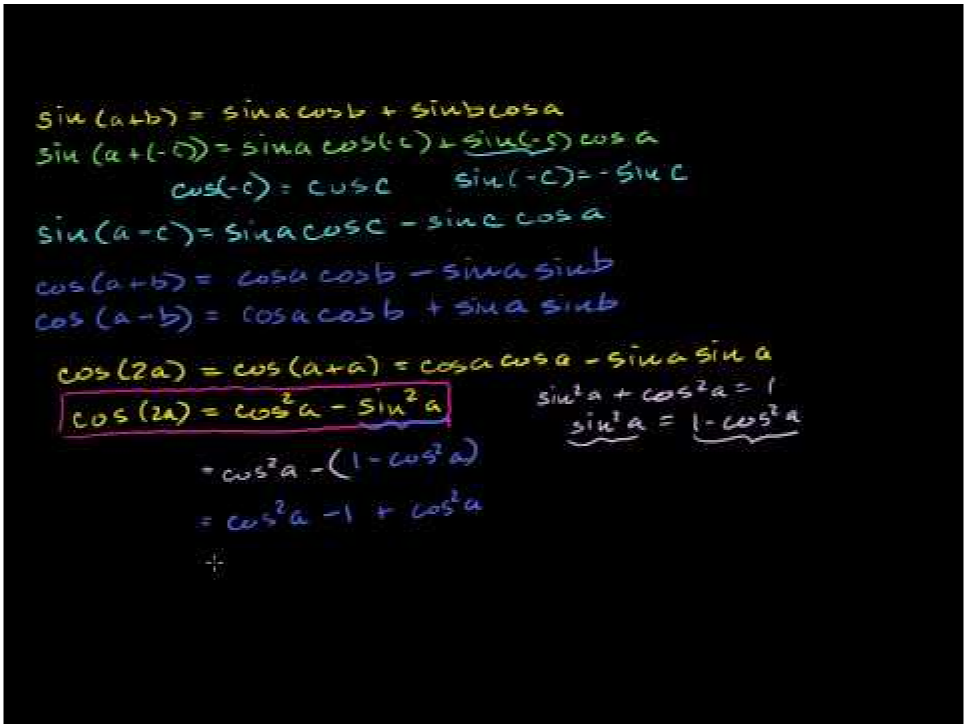
5) Trigonometric Identities



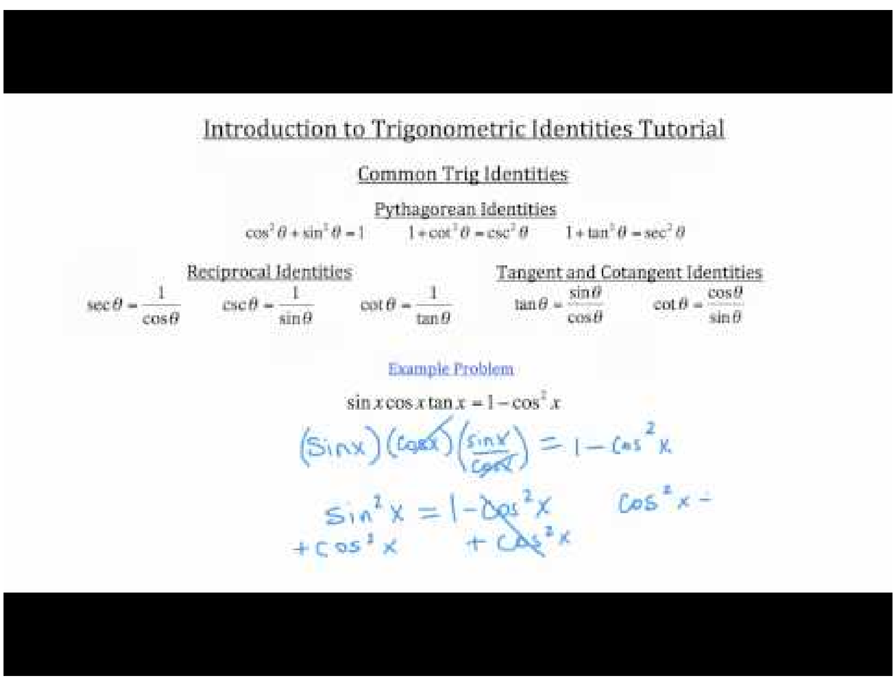
6) Verifying more difficult Trig. Identities



7) Review Trig identities (1)



Understanding Trig I



More on Trigonometry Identities

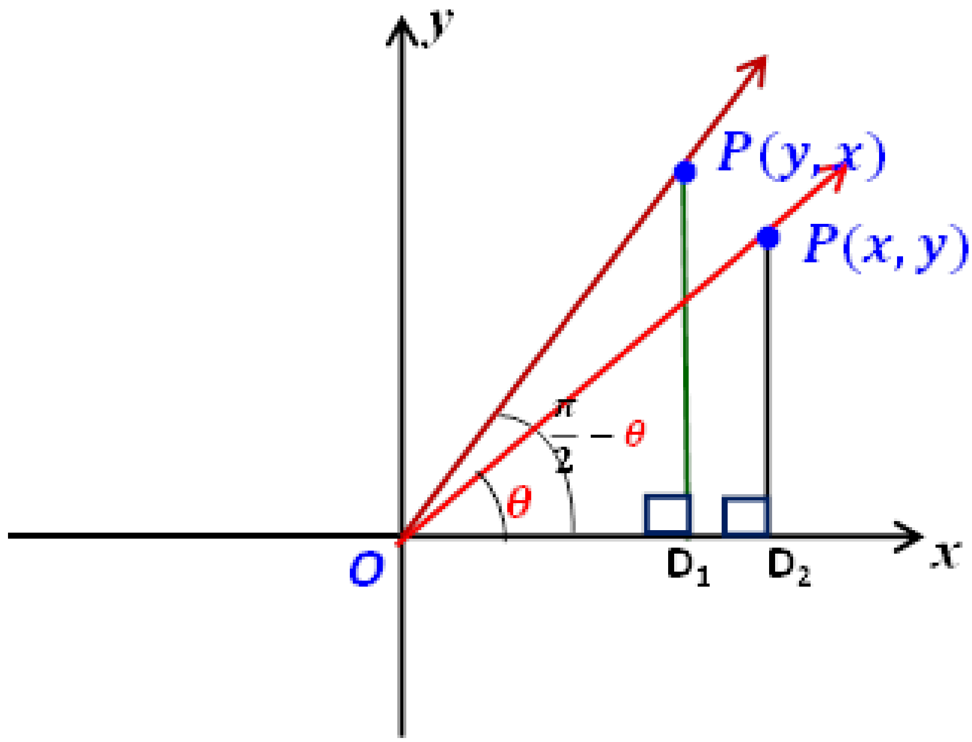
Further on Trigonometric Identities

a) Co-Function Identities

$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta,$ $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta,$

$\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta,$ $\csc\left(\frac{\pi}{2}-\theta\right)=\sec\theta,$

$\sec\left(\frac{\pi}{2}-\theta\right)=\csc\theta\,,$ $\cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta$



Proof: By definition, referring to the figure above

$$\sin\left(\frac{\pi}{2}-\theta\right)=\frac{x}{r}=\cos\theta$$

$$\cos\left(\frac{\pi}{2}-\theta\right)=\frac{y}{r}=\sin\theta$$

Similarly all the remaining cofunction identities follow

b) Even-Odd Identities

$\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$, $\tan(-\theta) = -\tan\theta$,

$\csc(-\theta) = -\csc\theta$, $\sec(-\theta) = \sec\theta$, $\cot(-\theta) = -\cot\theta$

Proof: Let θ be an angle of the 1st Quadrant,

then $-\theta$ is angle of 4th Quadrant, **see figure**.

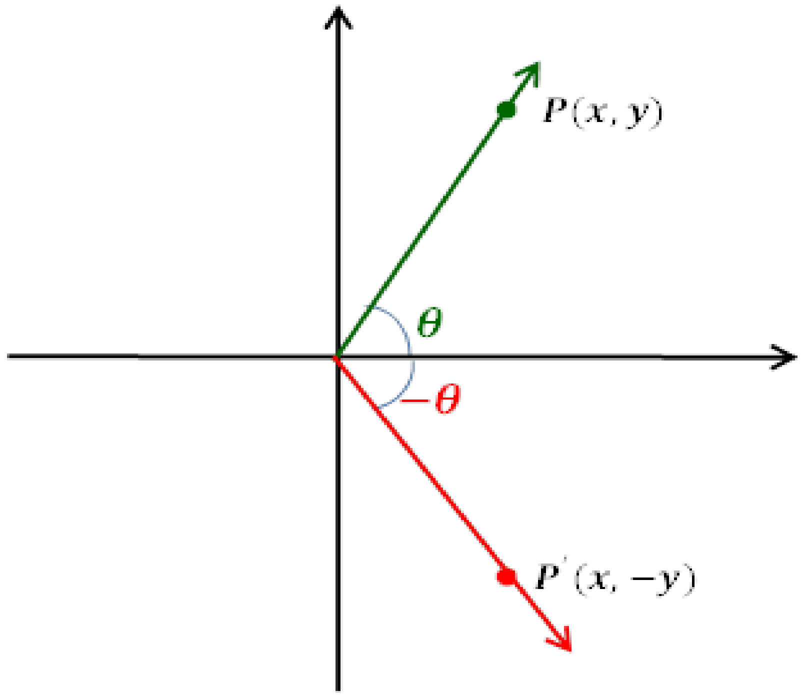
$r = OP = OP'$. Now,

$\sin(-\theta) = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta$

The rest of the Even – Odd Identities

for an angle of the 1st Quadrant can be

justified the same way.



4) Sum and difference formulas (page 71-73)

a) $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

b) $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

c) $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

d) $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

e) $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

f) $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

Example 1: Find the **exact** values of: a) $\sin 15^0$ **b)** $\cos 75^0$

Example 2: Simplify the following expression. $\sec\left(\frac{\pi}{2} - x\right) - \tan\left(\frac{\pi}{2} - x\right)\sin\left(\frac{\pi}{2} - x\right)$

Example: Book 2: Example 3.9, 3.10, 3.11, 3.12 reading (74 – 75)

(Book 2) Homework Exercises 3.2 page 76 –77: # 1 – 16 odd numbers

5) **Double-angle formulas (page 78)**

a) $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

b) $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

c) $\cos(2\theta) = 2\cos^2\theta - 1$

d) $\cos(2\theta) = 1 - 2\sin^2\theta$

e) $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

Proof: Proof follows from Sum Difference Formulas

Example: Book 2: Example 3.13, 3.14

Example 2: Given an angle for which $\sin(\alpha) = -3/5$ in Quadrant III, determine the values for $\sin(2\alpha)$, $\cos(2\alpha)$, $\tan(2\alpha)$, $\sin(\alpha/2)$, $\cos(\alpha/2)$, and $\tan(\alpha/2)$.

6) **Half-angle formulas (page 79 – 80)**

a) $\cos(\theta/2) = \pm\sqrt{\frac{1+\cos\theta}{2}}$

b) $\sin(\theta/2) = \pm\sqrt{\frac{1-\cos\theta}{2}}$

c) $\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

Example: Book 2: Example 3.15 page 81

P roof: Proof follows from Double-angle Formulas

Example 3: Find the Exact value of $\sin\left(\frac{\pi}{8}\right)$.

7) **Products as sums**

a) $\sin\alpha\cos\beta = ½[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

b) $\cos\alpha\sin\beta = ½[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

c) $\cos\alpha\cos\beta = ½[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$

d) $\sin\alpha\sin\beta = -½[\cos(\alpha+\beta) - \cos(\alpha-\beta)]$

8) **Sums as products**

a) $\sin A + \sin B = 2\sin ½(A+B)\cos ½(A-B)$

b) $\sin A - \sin B = 2\sin ½(A-B)\cos ½(A+B)$

c) $\cos A + \cos B = 2\cos ½(A+B)\cos ½(A-B)$

d) $\cos A - \cos B = -2\sin ½(A+B)\sin ½(A-B)$

(Book 2) Homework Exercises 3.3 page 81: # 1 – 18 odd numbers

Example 4: Show that $sec^2x + csc^2x = sec^2x\, csc^2x$

Example 5: Prove that

a) $\frac{\tan y}{\sin y} = \sec y$

b) $\sin y + \sin y \cot^2 y = \csc y$

c) $\tan x + \cot x = \sec x \csc x$

d) $\frac{1+\cos x}{\sin x} = \frac{\sin x}{1-\cos x}$

e) $\tan(\pi - x) = -tan\,(x)$

f) $\tan\left(\frac{3}{2}\pi + x\right) = -cot\,(x)$

Example 6: Use the basic trigonometric identities to determine the **other five values** of the trigonometric functions given that:

a) $\sin\alpha = 7/8$ and $\cos\alpha > 0$.

b) x is an angle in **quadrant III** and $sin\,x = -1/3$.

c) x is an angle in **quadrant IV** and **tan x = -5**.

d) x is in **quadrant II** and $sin\,x = 1/5$.

e) x is in **quadrant I** and $cot\,x = 1/5$.

Example 7: Write $A\sin bt + B\cos bt = a\sin(bt + c)$

Solution:

Solved Examples:

1) Simplify the following trigonometric expression. $csc\,(x)\, sin\,(\pi/2 - x)$

Solution:

- Use the identity $sin\,(\pi/2 - x) = cos(x)$ and simplify

  $csc\,(x)\, sin\,(\pi/2 - x) = csc\,(x)\, cos\,(x) = cot\,(x)$

2) Simplify the following trigonometric expression. $[sin^4x - cos^4x] / [sin^2x - cos^2x]$

Solution:

- Factor the denominator $[sin^4x - cos^4x] / [sin^2x - cos^2x]$

  $= [sin^2x - cos^2x][sin^2x + cos^2x] / [sin^2x - cos^2x]$

  $= [sin^2x + cos^2x] = 1$

3) Simplify the following trigonometric expression. $[sec(x)\, sin^2x] \,/\, [1 + sec(x)]$

Solution:

- Substitute sec (x) that is in the numerator by $\mathbf{1 \,/\, cos\,(x)}$ and simplify.
  $[sec(x)\, sin^2x] \,/\, [1 + sec(x)]$
  $= sin^2x \,/\, [\, cos\, x\, (1 + sec\,(x)\,]$
  $= sin^2x \,/\, [\, cos\, x + 1\,]$
- Substitute $\mathbf{sin^2x\ by\ 1 - cos^2x}$, factor and simplify.
  $= [\,1 - cos^2x\,] \,/\, [\,cos\, x + 1\,]$
  $= [\,(1 - cos\, x)(1 + cos\, x)\,] \,/\, [\,cos\, x + 1\,] = 1 - cos\, x$

4) Simplify the following trigonometric expression. $\mathbf{sin\,(-x)\, cos\,(\pi/2 - x)}$

Solution:

- Use the identities $\mathbf{sin\,(-x) = -\,sin\,(x)}$ and $\mathbf{cos\,(\pi \,/\, 2 - x) = sin\,(x)}$ and simplify

  $\mathbf{sin\,(-x)\, cos\,(\pi/\,2 - x) = -\,sin\,(x)\, sin\,(x) = -\,sin^2x}$

5) Simplify the following trigonometric expression. $\mathbf{sin^2x - cos^2x\, sin^2x}$

Solution:

- Factor $\mathbf{sin^2x}$ out, group and simplify $\mathbf{sin^2x - cos^2x\, sin^2x}$
  $= sin^2x\,(\,1 - cos^2x\,)$
  $= sin^4x$

6) Simplify the following trigonometric expression. $\mathbf{tan^4x + 2\,tan^2x + 1}$

Solution:

- Note that the given trigonometric expression can be written as a square
  $\mathbf{tan^4x + 2\,tan^2x + 1} = (\,\mathbf{tan^2x + 1})^{\mathbf{2}}$

- We now use the identity **1 + tan ^{2}x = sec ^{2}x**
  $\mathbf{tan^4x + 2\,tan^2x + 1} = (\,\mathbf{tan^2x + 1})^{\mathbf{2}}$ **= (sec ^{2}x) ^{2} = sec ^{4}x**

7) Add and simplify. $\mathbf{1 \,/\, [1 + cos\, x] + 1 \,/\, [1 - cos\, x]}$

Solution:

- In order to add the fractional trigonometric expressions, we need to have a common denominator
  $\mathbf{1 \,/\, [1 + cos\, x] + 1 \,/\, [1 - cos\, x]}$
  $= [\,1 - cos\, x + 1 + cos\, x\,] \,/\, [\,[1 + cos\, x]\,[1 - cos\, x]\,]$
  $= 2 \,/\, [1 - cos^2x]$
  $= 2 \,/\, sin^2x = 2\, csc^2x$

8) Write $\sqrt{(4 - 4\sin^2 x)}$ without square root for $(\pi/2) < x < \pi$.

Solution:

- Factor, and substitute $\mathbf{1 - \sin^2 x}$ **by** $\mathbf{\cos^2 x}$

$$\sqrt{(4 - 4\sin^2 x)} = \sqrt{4(1 - \sin^2 x)} = 2\sqrt{\cos^2 x} = 2\,|\cos(x)|$$

- Since $(\pi/2) < x < \pi$, $\cos x$ is less than zero and the given trigonometric expression simplifies to $= -2\cos(x)$

9) Simplify the following expression. $[1 - \sin^4 x] / [1 + \sin^2 x]$

Solution:

- Factor the denominator, and simplify

$$[1 - \sin^4 x] / [1 + \sin^2 x] = [1 - \sin^2 x]\,[1 + \sin^2 x] / [1 + \sin^2 x]$$

$$= [1 - \sin^2 x] = \cos^2 x$$

10) Add and simplify. $1 / [1 + \sin x] + 1 / [1 - \sin x]$

Solution :

- Use a common denominator to add

$$\frac{1}{[1 + \sin x]} + \frac{1}{[1 - \sin x]} = \frac{[1 - \sin x + 1 + \sin x]}{[(1 + \sin x)(1 - \sin x)]}$$

$$= 2 / [1 - \sin^2 x]$$

$$= 2 / \cos^2 x = 2\sec^2 x$$

11) Add and simplify. $\cos x - \cos x \sin^2 x$

Solution:

- factor $\cos x$ out ; $\cos x - \cos x \sin^2 x = \cos x\,(1 - \sin^2 x)$

$$= \cos x \cos^2 x$$

$$= \cos^3 x$$

12) Simplify the following expression. $\boldsymbol{tan^2x\, cos^2x + cot^2x\, sin^2x}$

Solution:

- Use the trigonometric identities $\boldsymbol{tan\, x = sin\, x \,/\, cos\, x}$ and $cot\, x = cos\, x \,/\, sin\, x$ to write the given expression as

  $\boldsymbol{tan^2x\, cos^2x + cot^2x\, sin^2x = (sin\, x \,/\, cos\, x)^2\, cos^2x + (cos\, x \,/\, sin\, x)^2\, sin^2x}$

- and simplify to get: $\boldsymbol{= sin^2x + cos^2x = 1}$

13) Simplify the following expression. $\boldsymbol{sec\left(\frac{\pi}{2} - x\right) - tan\left(\frac{\pi}{2} - x\right) sin\left(\frac{\pi}{2} - x\right)}$

Solution:

- Use the identities $\boldsymbol{sec\left(\frac{\pi}{2} - x\right) = csc\, x}$, $\boldsymbol{tan\left(\frac{\pi}{2} - x\right) = cot\, x}$ and $\boldsymbol{sin\left(\frac{\pi}{2} - x\right) = cos\, x}$ to write the given expression as

$$\boldsymbol{\sec\left(\frac{\pi}{2} - x\right) - \tan\left(\frac{\pi}{2} - x\right) \sin\left(\frac{\pi}{2} - x\right) = \csc x - \cot x \cos x}$$

$$\boldsymbol{= \csc x - \frac{\cos x}{\sin x}\, cos\, x}$$

$$\boldsymbol{= csc\, x - cos^2x/sin\, x}$$

$$\boldsymbol{= 1/sin\, x - cos^2x/sin\, x}$$

$$\boldsymbol{= (1 - cos^2x)\, /sin\, x}$$

$$\boldsymbol{= sin^2x/sin\, x}$$

$$\boldsymbol{= sin\, x}$$

14) Show that $\boldsymbol{cos^4x - sin^4x = \cos(2x)}$

Practice Problems